Function Spaces
Supplementary Exam Question Paper
Marks - 50, Duration - 3 Hours

1. [3+(2+2)+1 marks]
   (a) Write down the definition of compact metric space.
   (b) In $\mathbb{R}^n$ with Euclidean metric, provide examples of a compact and non-compact subsets of $\mathbb{R}^n$, respectively.
   (c) (True/False) Any non-trivial subspace of $\mathbb{R}^n$ is compact.

2. [(2+1)+(2+3) marks]
   (a) i. Write down the definition of complete metric space.
   ii. (True/False) For a compact metric space (M, d), $(C(M), \|.\|_\infty)$ is a complete metric space, where
   $$C(M) := \{f : M \to \mathbb{C} : f \text{ is continuous on } M, \|f\|_\infty = \sup_{x \in M} |f(x)| < \infty\}.$$
   (b) i. Write down the statement of Banachs contraction principle.
   ii. Illustrate the Banachs contraction principle by an example.

3. [2+1+2+3 marks]
   (a) Write down the definition of family of uniformly equicontinuous functions.
   (b) (True/False) For fixed $0 < \alpha \leq 1$ and $0 < K < \infty$, $\mathrm{Lip}_K\alpha$ is a family of uniformly equicontinuous functions, where
   $$\mathrm{Lip}_K\alpha := \big\{f : [0,1] \to \mathbb{C} : |f(x) - f(y)| \leq K|x - y|^\alpha,\ x, y \in [0,1]\big\}.$$
   (c) Write down the statement of Arzela-Ascoli theorem for $C(M)$.
   (d) Illustrate the Arzela-Ascoli theorem for $C(M)$ by an example.

4. [2+1+2+3 marks]
   (a) Write down the definition of subalgebra.
   (b) Give an example of subalgebra of $C(M)$.
   (c) Write down the statement of Stone-Weierstrass theorem for $C(M)$.
   (d) Illustrate the Stone-Weierstrass theorem for $C(M)$ by an example.

5. [2+2+4 marks]
   (a) Give an example of a function $f \in C^{2\pi}$, where $C^{2\pi}$ is the space of all 2π-periodic continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
   (b) Write down the Fourier series expansion of a function $f \in C^{2\pi}$.

(c) Prove that

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n}, \quad (-\pi < x < \pi).$$

6. [5+5 marks]

(a) Show that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges for all $|x| \leq 1$, but it does not converge uniformly on $|x| \leq 1$.

(b) If $f \in C[a,b]$, and $\int_a^b x^n f(x)dx = 0$ for all $n \geq 0$, then prove that $f = 0$.